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The Usual Behavior of Rational Approximation, II

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The speeds of convergence of best rational approximations, best polynomial approximations, and the modulus of continuity on the unit disc are compared. We show that, in a Baire category sense, it is expected that subsequences of these approximants will converge at the same rate. Similar problems on the interval $[-1, 1]$ are also examined. A problem raised by P. Turán (*J. Approx. Theory* **29**, 1980, 23–89) concerning rational approximation to non-analytically continuable f on the unit circle is negated as an application. © 1993 Academic Press, Inc.

1. INTRODUCTION

We examine questions concerning rational approximations to analytic functions in $|z| < 1$ and to continuous functions from the point of view of how they usually behave. As in [1], the notion of "*usually*" we adopt is the Baire categorical notion in a complete metric space.

Let A be the space of functions, which are analytic in $|z| < 1$ and continuous on $|z| = 1$, S the subset of A containing functions that can not be continued analytically beyond $|z| = 1$ at any point, $C_{2\pi}$ the class of continuous real functions of period 2π, and $C_{[-1,1]}$ the class of continuous real functions on $[-1, 1]$.

Write

$$E_n(f)_E = \min_{p \in \Pi_n} \|f - p\|_E = \min_{p \in \Pi_n} \max_{z \in E} |f(z) - p(z)|,$$

$$R_n(f)_E = \min_{r \in R_{n,n}} \|f - r\|_E,$$

$$\omega(f, t)_E = \max_{0 < |h| \leq t} \|f(z+h) - f(z)\|_E,$$

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where Π_n is the class of polynomials of degree at most n,

$$R_{n,n}=\left\{\frac{p}{q}: p, q \in \Pi_n\right\}.$$

Since Newman showed that $|x|$ is uniformly approximated by rationals much better than by polynomials (cf. [2]), substantial progress has been made in discovering classes of functions for which rational approximation is better than polynomial approximation. A well-known example is the Newman's Lip 1 conjecture which claims that

$$\lim_{n\to\infty} nR_n(f)_{[-1,1]}=0$$

whenever $f\in \text{Lip } 1$. This conjecture was proved by Popov [3].

In spite of these positive results, the present paper shows that, in a categorical sense, it is expected that subsequences of rational approximants and polynomial approximants will usually converge at the same rate. Similar type results for entire functions were considered in [1].

We adopt the familiar categorical vocabulary. A set B is "*nowhere dense*" if the interior of B closure is empty. A set B is "*category 1*" if B is a countable union of nowhere dense sets. A set B is "*residual*" if it is the complement of a category 1 set. So a residual set contains almost all functions from a Baire category point of view. A set B is a "G_δ" if it is a countable intersection of open sets.

2. Results for A

Let

$$\tilde{\Pi}_n = A\backslash \Pi_n,$$

and

$$D=\{z: |z|\leqslant 1\}.$$

In the sequel, we always write $p_n(f, z)$ to indicate the polynomial of best approximation to f of degree at most n, $r_n(f, z)$ a rational function of best approximation to f from $R_{n,n}$. Throughout the paper, the norms are the uniform norms.

THEOREM 1. *Let*

$$A_1=\left\{f\in A: \limsup_{n\to\infty} \frac{R_n(f)_D}{E_n(f)_D}=1\right\}.$$

Then A_1 is residual in A.

Proof. Let

$$A_1^n = \left\{ f \in A\text{: There is an } m_n \geqslant n \text{ such that } \frac{R_{m_n}(f)_D}{E_{m_n}(f)_D} > 1 - \frac{1}{n} \right.$$

$$\left. \text{and } E_{m_n}(f)_D \neq 0 \right\},$$

then

$$A_1 = \bigcap_{n=1}^{\infty} A_1^n \bigcap_{n=0}^{\infty} \tilde{\Pi}_n.$$

To show that A_1 is residual we need to show that each A_1^n is open and dense in A (note that each Π_n is closed and nowhere dense).

Let n be fixed. For any f, $g \in A$, we have

$$R_n(g)_D \geqslant \|f - r_n(g)\|_D - \|g - f\|_D \geqslant R_n(f)_D - \|f - g\|_D,$$

while

$$E_n(g)_D = \|g - p_n(g)\|_D \leqslant \|g - p_n(f)\|_D \leqslant E_n(f)_D + \|f - g\|_D,$$

and

$$E_n(g)_D \geqslant E_n(f)_D - \|f - g\|_D. \tag{1}$$

Thus

$$\frac{R_n(g)_D}{E_n(g)_D} \geqslant \frac{R_n(f)_D - \|f - g\|_D}{E_n(f)_D + \|f - g\|_D}. \tag{2}$$

If $f \in A_1^n$, then there is an $m_n \geqslant n$ such that

$$\frac{R_{m_n}(f)_D}{E_{m_n}(f)_D} > 1 - \frac{1}{n},$$

and

$$E_{m_n}(f)_D \neq 0.$$

From (1) and (2) we can deduce that there is a sufficiently small $\delta > 0$ such that for all $g \in A$ with $\|f - g\|_D \leqslant \delta$,

$$\frac{R_{m_n}(g)_D}{E_{m_n}(g)_D} > 1 - \frac{1}{n},$$

and

$$E_{m_n}(g)_D \neq 0.$$

So $g \in A_1^n$, which shows that each A_1^n is open.

Next, for any given $f \in A$ and ε, $0 < \varepsilon < 1/(2n)$, there is an $N > 0$ so that

$$\|f - p_N(f)\|_D < \frac{\varepsilon}{2}, \tag{3}$$

and

$$3^{-N} < \min\left\{\frac{\varepsilon}{2}, \frac{1}{2n}\right\}. \tag{4}$$

We observe that if $n \geqslant 2m+1$, then for any $q(z) \in \Pi_m$,

$$R_m(z^n + q(z))_D \geqslant 1. \tag{5}$$

In fact, suppose there is a rational $r(z) \in R_{m,m}$ such that

$$\|z^n + q(z) - r(z)\|_D < 1,$$

then by Rouché's theorem, $q(z) - r(z)$ has n zeros in D, but $q(z) - r(z)$ has numerator of degree less than n.

Now define

$$f^*(z) = p_N(f, z) + \sum_{j=N+1}^{\infty} m_j^{-1} z^{m_j},$$

where

$$m_j = 3^{j^j}.$$

By (3),

$$\|f - f^*\|_D \leqslant \|f - p_N(f)\|_D + \sum_{j=N+1}^{\infty} m_j^{-1} \|z^{m_j}\|_D \leqslant \frac{\varepsilon}{2} + m_N^{-1} < \varepsilon. \tag{6}$$

At the same time, for $k \geqslant N+1$,

$$E_{m_k}(f^*)_D \leqslant \sum_{j=k+1}^{\infty} m_j^{-1} \|z^{m_j}\|_D \leqslant m_{k+1}^{-1} + 2m_{k+2}^{-1}. \tag{7}$$

On the other hand, from (5),

$$\begin{aligned} R_{m_k}(f^*)_D &\geqslant \left\| m_{k+1}^{-1} z^{m_{k+1}} + p_N(f, z) + \sum_{j=N+1}^{k} m_j^{-1} z^{m_j} - r_{m_k}(f^*, z) \right\|_D \\ &\quad - \sum_{j=k+2}^{\infty} m_j^{-1} \|z^{m_j}\|_D \\ &\geqslant m_{k+1}^{-1} R_{m_k}\left(z^{m_{k+1}} + m_{k+1}\Big(p_N(f, z) + \sum_{j=N+1}^{k} m_j^{-1} z^{m_j}\Big)\right)_D - 2m_{k+2}^{-1} \\ &\geqslant m_{k+1}^{-1} - 2m_{k+2}^{-1}. \end{aligned}$$

So together with (4) and (7) we get

$$\frac{R_{m_k}(f^*)_D}{E_{m_k}(f^*)_D} \geqslant \frac{m_{k+1}^{-1} - 2m_{k+2}^{-1}}{m_{k+1}^{-1} + 2m_{k+2}^{-1}} > 1 - \frac{1}{n},$$

that is, $f^* \in A_1^n$. This combines with (6) to prove that each A_1^n is open and dense in A and completes Theorem 1. ▮

THEOREM 2. *Let*[1]

$$A_2 = \left\{ f \in A\text{: there is a sequence } \{n_k\} \text{ such that } \lim_{k \to \infty} \frac{R_{n_k}(f)_D}{\omega(f, n_k^{-1})_D} = \frac{1}{2} \right\}.$$

Then A_2 is residual in A.

Proof. Let

$$A_2^n = \left\{ f \in A\text{: there is an } m_n \geqslant n \text{ such that } \frac{1}{2} + \frac{1}{n} > \frac{R_{m_n}(f)_D}{\omega(f, m_n^{-1})_D} > \frac{1}{2} - \frac{1}{n} \right\},$$

then

$$A_2 = \bigcap_{n=1}^{\infty} A_2^n.$$

In a manner similar to the proof of Theorem 1, we can prove that each A_2^n is open. Now let n be fixed. For any given $f \in A$ and ε, $0 < \varepsilon < 1/2n$, there is an $N > 0$ such that

$$\|f - p_N(f)\|_D < \min\left\{\frac{\varepsilon}{2}, \|f\|_D\right\} \tag{8}$$

and

$$3^{-N} < \min\left\{\frac{\varepsilon}{2}, \frac{1}{2n}\right\}.$$

Set

$$f^*(z) = p_N(f, z) + \sum_{j=N+1}^{\infty} m_j^{-1} z^{m_j^2},$$

where

$$m_j = 3^{j!}.$$

[1] The constant 1/2 in the following definition can be replaced by any constant c, $0 < c \leqslant 1/2$. Corresponding variation of the constants in Theorems 3, 5, 6, 8, and 9 is also allowed.

As in the proof of Theorem 1, we have

$$\|f-f^*\|_D<\varepsilon$$

and for $k\geqslant N+1$,

$$R_{n_k}(f^*)_D\geqslant m_k^{-1}-2m_{k+1}^{-1},$$

where $n_k=(m_k^2-1)/2-1$. Meanwhile, by Bernstein's inequality and (8),

$$\begin{aligned}\omega(f^*, n_k^{-1})_D &\leqslant\left(\|p'_N(f)\|_D+\sum_{j=N+1}^{k} m_j\right)n_k^{-1}+2\sum_{j=k+1}^{\infty} m_j^{-1}\\ &\leqslant\left(2N\|f\|_D+\sum_{j=N+1}^{k-1} m_j\right)n_k^{-1}+2m_k^{-1}+4m_{k+1}^{-1}\\ &\leqslant 2m_k^{-1}+o(m_k^{-1}),\qquad k\to\infty.\end{aligned}$$

Therefore for sufficiently large k,

$$\frac{R_{n_k}(f^*)_D}{\omega(f^*, n_k^{-1})_D}>\frac{1}{2}-\frac{1}{n}.$$

A similar calculation shows that

$$R_{n_k}(f^*)_D\leqslant m_k^{-1}+2m_{k+1}^{-1},$$

and

$$\omega(f^*, n_k^{-1})_D\geqslant 2m_k^{-1}-o(m_k^{-1}),\qquad k\to\infty.$$

So for sufficiently large k,

$$\frac{R_{n_k}(f^*)_D}{\omega(f^*, n_k^{-1})_D}<\frac{1}{2}+\frac{1}{n},$$

that is, $f^*\in A_2^n$. This finishes the proof. ▮

Similarly, we can prove

THEOREM 3. *Let*

$$A_3=\left\{f\in A: \text{there is a sequence } \{n_k\} \text{ such that } \lim_{k\to\infty}\frac{E_{n_k}(f)_D}{\omega(f, n_k^{-1})_D}=1\right\}.$$

Then A_3 is residual in A.

3. Results for $C_{2\pi}$ and $C_{[-1,1]}$

In $C_{2\pi}$ and $C_{[-1,1]}$, there are corresponding results.

THEOREM 4. *Let*

$$B_1 = \left\{ f \in C_{2\pi} : \limsup_{n \to \infty} \frac{R_n(f)_{[0,2\pi]}}{E_n(f)_{[0,2\pi]}} = 1 \right\}.$$

Then B_1 is residual in $C_{2\pi}$.

THEOREM 5. *Let*

$$B_2 = \left\{ f \in C_{2\pi} : \text{there is a sequence } \{n_k\} \text{ such that } \lim_{k \to \infty} \frac{R_{n_k}(f)_{[0,2\pi]}}{\omega(f, n_k^{-1})_{[0,2\pi]}} = \frac{1}{2} \right\}.$$

Then B_2 is residual in $C_{2\pi}$.

THEOREM 6. *Let*

$$B_3 = \left\{ f \in C_{2\pi} : \text{there is a sequence } \{n_k\} \text{ such that } \lim_{k \to \infty} \frac{E_{n_k}(f)_{[0,2\pi]}}{\omega(f, n_k^{-1})_{[0,2\pi]}} = 1 \right\}.$$

Then B_3 is residual in $C_{2\pi}$.

THEOREM 7. *Let*

$$C_1 = \left\{ f \in C_{[-1,1]} : \limsup_{n \to \infty} \frac{R_n(f)_{[-1,1]}}{E_n(f)_{[-1,1]}} = 1 \right\}.$$

Then C_1 is residual in $C_{[-1,1]}$.

THEOREM 8. *Let*

$$C_2 = \left\{ f \in C_{[-1,1]} : \text{there is a sequence } \{n_k\} \text{ such that } \lim_{k \to \infty} \frac{R_{n_k}(f)_{[-1,1]}}{\omega(f, n_k^{-1})_{[-1,1]}} = \frac{1}{2} \right\}.$$

Then C_2 is residual in $C_{[-1,1]}$.

THEOREM 9. *Let*

$$C_3 = \left\{ f \in C_{[-1,1]} : \text{there is a sequence } \{n_k\} \text{ such that } \lim_{k \to \infty} \frac{E_{n_k}(f)_{[-1,1]}}{\omega(f, n_k^{-1})_{[-1,1]}} = 1 \right\}.$$

Then C_3 is residual in $C_{[-1,1]}$.

All the proofs of these theorems are quite similar to those of Theorems 1 and 2. However, there are some differences in the space $C_{[-1,1]}$, so we will discuss Theorem 8.

Proof of Theorem 8. The only thing different from the proof of Theorem 2 is the construction of a function

$$f^*(x) \in \left\{ f \in C_{[-1,1]}\text{: there is an } m_n \geq n \text{ such that } \frac{1}{2}+\frac{1}{n} > \frac{R_{m_n}(f)_{[-1,1]}}{\omega(f, m_n^{-1})_{[-1,1]}} > \frac{1}{2}-\frac{1}{n} \right\}$$

with

$$\|f - f^*\| < \varepsilon$$

for any given $f \in C_{[-1,1]}$, $n \geq 1$, and $\varepsilon > 0$. That is done by constructing

$$f^*(x) = p_N(f, x) + \sum_{j=N+1}^{\infty} m_j^{-1} T_{m_j^2}\left(\frac{x}{2}\right)$$

for sufficiently large N, where

$$T_n(x) = \cos(n \arccos x),$$

$$m_j = 9^{j^j}, \qquad n_k = [\tfrac{1}{12} m_k^2 - 1].$$

The rest of the proof remains almost the same as that of Theorem 2, and we omit the details. ∎

4. A Problem of Turán

Turán raised the following problem in his well-known "problem-paper" [5].

Problem LXXXVI. *Is it true that, for $f \in S$, we have*

$$|f(z) - R_n^*(z)| = o(1)\, \omega(f, n^{-1})_D, \qquad |z| \leq 1$$

with a suitable rational function R_n^*?

We show that its answer is negative.

Let X_n be the subset of $C_{2\pi}$ consisting of those functions f for which there exists a $t \in [0, 2\pi)$ such that

$$|f(s) - f(t)| \leq n\, |s - t|$$

for all $s \in [0, 2\pi)$.

LEMMA 1. *Fix n. Let $f \in C_{2\pi}$ have a continuous derivative. Given an $\varepsilon > 0$, then there exists a function $L(f) \in C_{2\pi}$ as well as $\tilde{L}(f) \in C_{2\pi}$, a $\delta > 0$, and a constant $M > 0$ so that*

$$\|f - L(f)\|_{[0,2\pi]} < \varepsilon, \tag{9}$$

$$\|\tilde{f} - \tilde{L}(f)\|_{[0,2\pi]} < Mn \, \|f'\|_{[0,2\pi]} \sqrt{\varepsilon}, \tag{10}$$

and for all $g \in C_{2\pi}$ with $\|L(f) - g\|_{[0,2\pi]} < \delta/(4(n+1))$, we have

$$g \notin X_n,$$

where

$$\tilde{f} = -\frac{1}{\pi} \int_0^\pi \frac{f(x+t) - f(x-t)}{2 \tan t/2} \, dt$$

is the conjugate function of f.

Proof. Since f has a continuous derivative,

$$|f(x_1) - f(x_2)| \leqslant \|f'\|_{[0,2\pi]} \, |x_1 - x_2|.$$

Take

$$\delta = \delta(\varepsilon) = \max\left\{\frac{\pi}{N} : \frac{\pi}{N} \leqslant \frac{\varepsilon}{8} \min\{1/\|f'\|_{[0,2\pi]}, 1\}, N = 1, 2, ...\right\}.$$

Set

$$x_k = \frac{k\delta}{n+1}, \qquad k = 0, 1, ..., 2\frac{\pi}{\delta}(n+1).$$

Define

$$L(f, x) = (n+1)\,\delta^{-1}\left(f(x_k) - f(x_{k-1}) + (-1)^k \frac{\varepsilon}{2}\right)(x - x_{k-1}) + f(x_{k-1})$$

$$+ (-1)^{k-1}\frac{\varepsilon}{4}, \qquad x \in [x_{k-1}, x_k), \qquad k = 1, 2, ..., 2\frac{\pi}{\delta}(n+1).$$

Direct calculation now leads to (9). Also,

$$\tilde{f}(x) - \tilde{L}(f, x)$$
$$= -\frac{1}{\pi} \int_0^\pi \frac{f(x+t) - f(x-t) - L(f, x+t) + L(f, x-t)}{2 \tan t/2} \, dt$$

$$= -\frac{1}{\pi}\left(\int_0^{\sqrt{\varepsilon}} \frac{f(x+t)-f(x-t)}{2\tan t/2}\,dt - \int_0^{\sqrt{\varepsilon}} \frac{L(f,x+t)-L(f,x-t)}{2\tan t/2}\,dt\right.$$

$$\left.+\int_{\sqrt{\varepsilon}}^{\pi} \frac{f(x+t)-L(f,x+t)-f(x-t)+L(f,x-t)}{2\tan t/2}\,dt\right)$$

$$=: -\frac{1}{\pi}(\Sigma_1+\Sigma_2+\Sigma_3).$$

We check that

$$|\Sigma_1| = O(\|f'\|_{[0,2\pi]}\sqrt{\varepsilon}),$$
$$|\Sigma_2| = O(n\,\|f'\|_{[0,2\pi]}\sqrt{\varepsilon}),$$

while from (9),

$$|\Sigma_3| = O(\sqrt{\varepsilon}^{\,-1}\varepsilon) = O(\sqrt{\varepsilon}),$$

and (10) now follows from combining all these estimates.

On the other hand, for any $x \in [0, 2\pi)$, say, $x \in [x_k, x_{k+1})$, there is a $t_x = x+\delta/(2(n+1))$ or $t_x = x-\delta/(2(n+1))$, according to the cases $x \in [x_k, (x_k+x_{k+1})/2)$ or $x \in [(x_k+x_{k+1})/2, x_{k+1})$, respectively, such that

$$|L(f,x)-L(f,t_x)| \geqslant \frac{\varepsilon}{8}(n+1)\,\delta^{-1}\,|x-t_x| \geqslant (n+1)\,|x-t_x|.$$

Now for all $g \in C_{2\pi}$ with $\|L(f)-g\|_{[0,2\pi]} < \delta/(4(n+1))$, the above estimate gives

$$|g(x)-g(t_x)| \geqslant |L(f,x)-L(f,t_x)| - 2\,\|L(f)-g\|_{[0,2\pi]}$$
$$> (n+1)\,|x-t_x| - |x-t_x| = n\,|x-t_x|,$$

so $g \notin X_n$. Lemma 1 is therefore proved. ∎

LEMMA 2. *There exists a dense G_δ in S.*

Proof. The class of all polynomials with rational coefficients, $T \subset A$, is countable. We write all elements of T as $t_1(z), t_2(z), \ldots$. We see from Lemma 1 that, for any given m, n, and k, there is a function $L_{kmn}(\theta) \in C_{2\pi}$ as well as $\tilde{L}_{kmn}(\theta) \in C_{2\pi}$ such that

$$|L_{kmn}(\theta) - \operatorname{Re}(t_m(e^{i\theta}))| < \frac{1}{k}, \tag{11}$$

and

$$|\tilde{L}_{kmn}(\theta) - \operatorname{Im}(t_m(e^{i\theta}))| = |\tilde{L}_{kmn}(\theta) - \tilde{\operatorname{Re}}(t_m(e^{i\theta}))| < \frac{Mn\,\|t_m'\|_{[0,2\pi]}}{\sqrt{k}}. \tag{12}$$

Moreover, the $(\delta(k^{-1})/(4(n+1)))$-neighbourhood of $L_{kmn}(\theta)$, N_{kmn}, does not intersect X_n. Write

$$f^*(\xi)=f(\theta)$$

for $\xi=e^{i\theta}$. So $f^*(\xi)$ is clearly a continuous function with respect to ξ if $f\in C_{2\pi}$. Now set

$$U(f,z)=\frac{1}{2\pi i}\int_{|\omega|=1} f^*(\omega)\frac{\omega+z}{\omega-z}\frac{d\omega}{\omega}, \qquad |z|<1,$$

then $U(f,z)$ is an analytic function in $|z|<1$, and

$$\lim_{z\to\xi}\operatorname{Re}(U(f,z))=f(\theta),$$

$$\lim_{z\to\xi}\operatorname{Im}(U(f,z))=\tilde{f}(\theta).$$

So with (11) and (12) we get

$$\|t_m(z)-U(L_{kmn},z)\|_D\leqslant 2Mn\,\|t_m'\|_{[0,2\pi]}\sqrt{k}^{-1}. \tag{13}$$

Let

$$N^*_{kmn}=\left\{f\in A\colon \|f-U(L_{kmn})\|_D<\frac{\delta(k^{-1})}{4(n+1)}\right\},$$

$$G_n=\bigcup_{m=1}^{\infty}\bigcup_{k=1}^{\infty}N^*_{kmn}.$$

It is evident that N^*_{kmn} does not intersect the set

$$X_n^*=\{f\in A\colon \operatorname{Re}(f(e^{i\theta}))\in X_n\}$$

since N_{kmn} does not intersect X_n. Furthermore, since T is dense in A and G_n contains all functions $U(L_{kmn},z)$ for $k,m=1,2,\ldots$, together with (13), we see that G_n is dense in A. Define

$$G=\bigcap_{n=1}^{\infty}G_n,$$

then G is a dense G_δ in A and clearly, G only contains functions whose real parts are nowhere differentiable on $|z|=1$, that is, $G\subset S$. ▮

We conclude that Turán's problem has a negative answer.

THEOREM 10. *There exist functions $f\in S\cap A_2$.*

Proof. From Lemma 2 and Theorem 2, a subset of such f is residual. ∎

However, much as in Theorem 2, we can establish the following weak form of Turán's problem.

THEOREM 11. *Let*

$$A_4 = \left\{ f \in A : \limsup_{n \to \infty} \frac{R_n(f)_D}{\omega(f, n^{-1})_D} = 0 \right\}.$$

Then A_4 is residual in A.

Similarly, in the notation of Theorems 5 and 8 we have

THEOREM 12. *There exist functions $f \in B_2$ which are nowhere differentiable.*

THEOREM 13. *There exist functions $f \in C_2$ which are nowhere differentiable.*

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